Short Trees - Solution

Given are $n, k \geq 1$, we will now define a sequence $(N_\ell)_{\ell \in \mathbb{Z}_+}$ on the positive integers, with N_ℓ equal to the number of paths of length ℓ in $G_{n,k}$. Note that

$$N_1 = n,$$

because the root node has n neighbours.

We will now set up a recursive for N_ℓ, distinguishing the cases where ℓ is odd and even. We distinguish between these two cases because in our problem paths of even length end in the root or leaf nodes and paths of odd length end in the middle nodes.

In the case where ℓ is even we have $\ell = 2l$, with $l \in \mathbb{Z}_+$. We now want to count the number of ways we can add 1 step to the paths of length $\ell - 1$. Note that the paths of length $\ell - 1 = 2l - 1$ are odd and so they end in the middle nodes. Each middle node has $k + 1$ neighbours namely its leafs and the root node, so we get that

$$N_{2l} = (k+1)N_{2l-1}.$$

Now we look at the case where $\ell > 1$ is odd, we then have $\ell = 2l + 1$ for some $l \in \mathbb{Z}_+$. Note that we want to know the number of ways we can extend the paths of length $\ell - 1 = 2l$ by 1 step. Since $2l$ is even we know that the paths ended in either the root node or one of the leaf nodes. If we ended in one of the leaf nodes we have exactly one choice to extend our path, namely to go back to its middle node. If we ended in the root node we can extend the path by choosing one out of the n neighbours, so we can write

$$N_{2l+1} = nN_{2l}^{\text{root}} + N_{2l}^{\text{leaf}},$$

where N_{2l}^{leaf} is the number of paths of length $2l$ that end in a leaf node and N_{2l}^{root} is the same, but for the root node.

The paths of length $2l$ where created by choosing one of the k leafs of 1 root node neighbouring the middle node, so we have $N_{2l}^{\text{leaf}} = kN_{2l-1}$ and $N_{2l}^{\text{root}} = N_{2l-1}$, so we can conclude that

$$N_{2l+1} = nN_{2l-1} + kN_{2l-1} = (n+k)N_{2(l-1)+1}.$$

One can now reduce these recurrence relations to the following direct formulas using induction, such that for all $l \in \mathbb{N}$,

$$\begin{aligned} N_{2l+1} &= \qquad n(n+k)^l \\ N_{2l+2} &= (k+1)n(n+k)^l. \end{aligned}$$